

CHAPTER 18 -- MAGNETIC FIELDS

18.1) A currentless wire is electrically neutral (i.e., its positive and negative charge densities are equal). Yet when electrons flow and length contraction happens, we still assume that the wire is electrically neutral in the lab frame. How can both be the case?

When current flows, a battery must be providing charge to the wire at a certain rate. What really happens is that the rate of flow of electrons onto the wire from the battery couples with the electron's length contraction to keep the electron density constant and equal to the proton density.

18.2) To determine the direction of a magnetic field produced by a current-carrying wire at some point, think of a circle centered on the wire that passes through the point of interest. The field will be tangent to the circle oriented clockwise or counterclockwise as defined by the right-thumb rule (the right thumb rule: point the right thumb in direction of current--your fingers will curl in direction of B).

With all that in mind, Figure 1a shows the circulation of the magnetic field set up by the current in wire 1, as identified in the sketch.

Notice that the direction of the B-field at Points E is upward and to the right.

Figure 1b shows the circulation of the magnetic field set up by the current in wire 2. Notice that the direction of the B-field at Points E is downward and to the right.

How does wire 1's field at Point E superimpose (i.e., add vectorially) with wire 2's field at that point? The y components add to zero leaving only a net x component to the right. So what is the trend for all of the wires? The net field will be to the right.

In short, to determine the net magnetic field direction for a group of wires, use the right-thumb rule on selected wires in the configuration to get a sense of the general trends, then pray for symmetry. With luck, components will cancel and you will be left with a clear-cut direction.

a.) At Point A: According to the right-thumb rule, the wires in the upper row and to the left of Point A will generate B-fields that will have components in the +x and +y directions (see Figure 1a). The upper-row

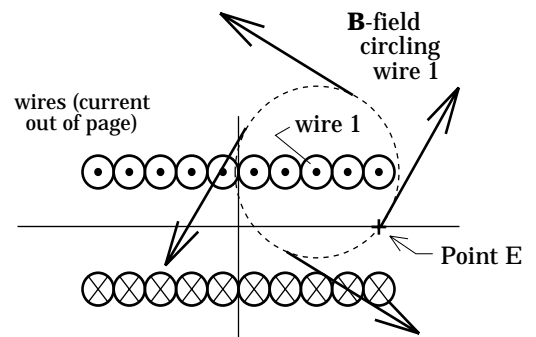


FIGURE 1a

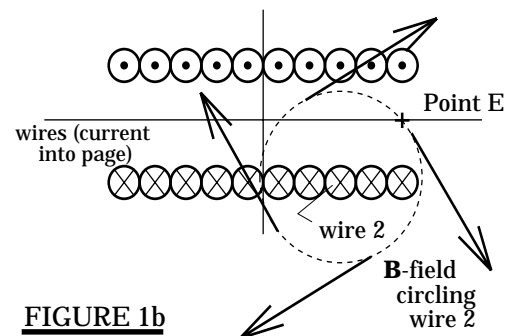


FIGURE 1b

wires to the right of Point A will generate B-field components in the +x and -y directions. Due to symmetry, the y components will add to zero and the net effect from the upper row will be a B-field in the +x direction.

The wires in the lower row and to the left of Point A will generate B-fields that will have components along the +x and -y directions. The lower row wires to the right of Point A will generate B-field components in the +x and +y directions (see Figure 1b). Again, due to symmetry, the y components will add to zero and the net effect from the lower row will be a B-field in the +x direction.

Bottom line: The net B-field will be in the +x direction.

b.) At Point E: The B-field generated by the upper row of wires will have components along the +x and +y axes. The B-field generated by the lower row of wires will have components along the x and -y directions.

Due to symmetry, the y components will add to zero leaving a net B-field in the +x direction.

c.) At Point C: The upper row of wires will all have B-field components in the -x direction with the left half having +y components while the right half has -y components (the y components add to zero). A similar y component situation exists with the lower row with the x component of their B-field in the +x direction. As the upper row is closer to Point C, the net field will be in the -x direction.

d.) At Point D: This is similar to Point C except Point D is far away. As you get further and further away, the magnitude of the opposing fields gets closer and closer while additionally getting smaller and smaller. For all intents and purposes, the components of the net B-field far away will add to zero.

e.) A charge placed at Point E will feel NO FORCE. Why? Because static magnetic fields can only change the direction of a charge's motion. They are NOT able to accelerate a charge in the sense of changing the magnitude of the charge's velocity.

18.3) The cross product in $F = qv \times B$ allows us to determine the DIRECTION of the force on a positive charge moving in a B-field. Using the right hand rule ($v \times B$) on a charge moving in the figure, we will see positive charge bend in the direction of that force while negative charge bends opposite that direction. Noting that non-charged particles do not deviate in the B-field at all, we can write:

- a.) --Charges A and G must be positive;
- Charges D and E must be negative;
- Charges C and F must be neutral.

b.) Assuming v and B are perpendicular to one another (i.e., the sine of the angle between the two vectors is one), the magnitude of the magnetic force will be $F = qvB$. Noting that all magnetic forces are centripetal in nature, we can use N.S.L. to write:

$$\begin{aligned}\underline{\Sigma F}_{\text{cent:}} \\ qvB &= ma_c \\ &= m(v^2/R)\end{aligned}$$

or

$$v = qBR/m.$$

In other words, for a fixed q , B , and m , the velocity is proportional to the radius of the motion. The largest radius in the picture appears to be that associated with charge G (charge D is a close second), therefore charge G is moving the fastest.

c.) Re-manipulating the previous N.S.L. equation above, we get:

$$q = mv/BR.$$

For a given m , v , and B , the charge is indirectly proportional to the radius of motion. In other words, holding all else constant, a large charge will have a small radius (this makes sense--the bigger the charge, the bigger the magnetic force on the particle and, hence, the tighter the circle). As such, the greatest charge should be charge A.

18.4) This requires the use of $F = qvB \sin \theta$. Expanding to determine the magnitude of the force, we get:

$$\begin{aligned}F &= q v B \sin \theta \\ (1.7 \times 10^{-8} \text{ nt}) &= (5 \times 10^{-10} \text{ C})(3 \times 10^2 \text{ m/s})(.138 \text{ T}) \sin \theta \\ \Rightarrow \theta &= 55.2^\circ.\end{aligned}$$

18.5) We will begin with the centripetal aspect of magnetic fields:

a.) Using the magnetic force equation:

$$\begin{aligned}\underline{\Sigma F}_{\text{cent:}} \\ q v B \sin \theta &= m a_c \\ &= m(v^2/R).\end{aligned}$$

Manipulating yields:

$$B = mv/[qR \sin \theta].$$

Assuming the angle is at 90° (to optimize the field's strength), we get:

$$\begin{aligned} B &= mv/[qR] \\ &= (6.67 \times 10^{-27} \text{ kg}) [.95(3 \times 10^8 \text{ m/s})] / [(1.6 \times 10^{-19} \text{ C})(100 \text{ m})] \\ &= .119 \text{ Teslas.} \end{aligned}$$

Note: The earth's magnetic field strength is 6×10^{-5} teslas. That means the accelerator will be using a B-field that is approximately 2000 times as large as the earth's field. (This is important as the earth's field cannot be screened out and, hence, cannot be ignored when the required field is small).

b.) The magnetic field relationship derived above still holds. That is:

$$B = mv/(qR).$$

In that expression, the mass m , the charge q , and the track radius R are all constant. If we can derive an expression for the appropriate velocity as a function of time, we will have the magnetic field as a function of time.

i.) To begin, consider the forces acting on the charge as it picks up speed around the track. Specifically, consider what happens when the charge accelerates between two plates. We know the electric force produced by the electric field acting on the charge is the accelerating factor in the problem. That means N.S.L. yields:

$$F_{\text{elect}} = m(dv/dt),$$

where the acceleration variable has been written in terms of the velocity's time-derivative.

ii.) As has been mentioned, this force is produced by the electric field between the plates. The relationship between an electric force and its associated electric field is:

$$F_{\text{elect}} = qE,$$

where q is the charge being accelerated and E is the electric field between the plates.

iii.) The electric field is present due to the electrical potential difference between the plates. The relationship between an electric field and its associated electrical potential field is:

$$\Delta V = -E \cdot d,$$

where the magnitude of d equals the distance L between the plates in this case. As the positive charges accelerate from higher to lower electrical potential (i.e., from V_{plate} to zero), we can rewrite this as:

$$(0 - V_{\text{plate}}) = -EL \cos 0^\circ,$$

or

$$E = V_{\text{plate}}/L.$$

iv.) With this, we can rewrite N.S.L. as:

$$\begin{aligned} qE &= m \frac{dv}{dt} \\ \Rightarrow q \frac{V_{\text{plate}}}{L} &= m \frac{dv}{dt} \\ \Rightarrow dv &= \frac{qV_{\text{plate}}}{mL} dt \\ \Rightarrow \int_{v=0}^v dv &= \frac{qV_{\text{plate}}}{mL} \int_{t=0}^t dt \\ \Rightarrow v &= \frac{qV_{\text{plate}}}{mL} t. \end{aligned}$$

v.) Knowing $v(t)$, we can write $B(t)$. Doing so yields:

$$\begin{aligned} B &= \frac{mv}{qR} \\ &= \frac{m \left[\frac{qV_{\text{plate}}}{mL} t \right]}{qR} \\ &= \frac{V_{\text{plate}}}{LR} t. \end{aligned}$$

c.) At $.95c$, the particle's mass is:

$$m_{\text{rel}} = m_{\text{rest}} / [1 - (v/c)^2]^{1/2}$$

$$m_{\text{rel}} = (6.67 \times 10^{-27} \text{ kg}) / [1 - [(0.95)(3 \times 10^8 \text{ m/s}) / (3 \times 10^8 \text{ m/s})]^2]^{1/2}$$

$$= 2.14 \times 10^{-26} \text{ kgs.}$$

With this, Part a becomes:

$$B = m_{\text{rel}} v / (qR),$$

$$= (2.14 \times 10^{-26} \text{ kg}) [0.95(3 \times 10^8 \text{ m/s})] / [(1.6 \times 10^{-19} \text{ C})(100 \text{ m})]$$

$$= .381 \text{ Teslas.}$$

Conclusion: The B-field has to be over three times as large to accommodate for relativistic effects. That would produce a whopping big error if not taken into account!

18.6)

a.) The fact that the charge is positive coupled with the cross product $F = qv \times B$ allows us to draw the path of the particle as it moves through the B-field (see Figure 2 to the right).

b.) This is a conservation of energy problem (you have a charge "falling" through a potential difference). As such:

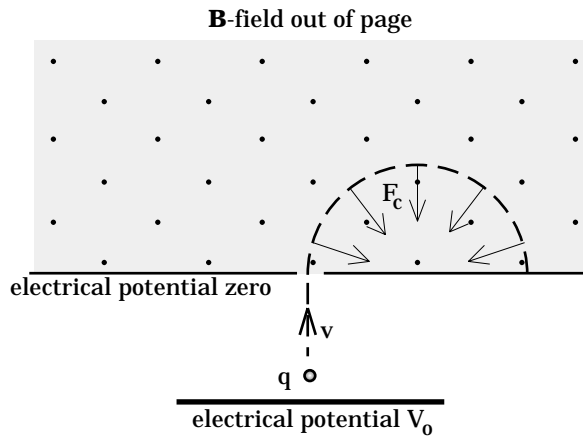


FIGURE 2

$$\Sigma KE_1 + \Sigma U_1 + \Sigma W = \Sigma KE_2 + \Sigma U_2$$

$$0 + qV_0 + 0 = (1/2)mv^2 + 0$$

$$\Rightarrow \Rightarrow v = [2 q V_0 / m]^{1/2}$$

$$= [2(4 \times 10^{-9} \text{ C})(2000 \text{ V}) / (5 \times 10^{-16} \text{ kg})]^{1/2}$$

$$= 1.79 \times 10^5 \text{ m/s.}$$

c.) The centripetal aspect of the magnetic force (assuming the angle between B and v is a right angle) yields:

$$\begin{aligned}
 \Sigma F_{\text{cent}}: \\
 q v B \sin \theta &= m a_c \\
 &= m(v^2/R) \\
 \Rightarrow R &= mv/qB \\
 &= (5 \times 10^{-16} \text{ kg})(1.79 \times 10^5 \text{ m/s}) / (4 \times 10^{-9} \text{ C})(1.8 \text{ T}) \\
 &= .0124 \text{ meters.}
 \end{aligned}$$

18.7) This is a problem in which there exists both an unknown magnetic field and a known electric field coexisting in the same space (a sketch of the known information is shown in Figure 3). As such, we need to use Lorentz's equation (Lorentz's equation incorporates all possible electric-borne forces into one expression), or:

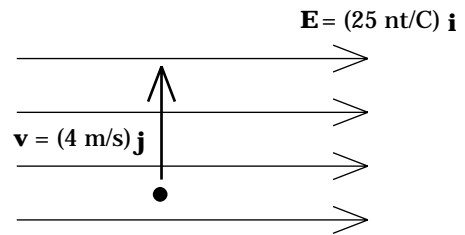


FIGURE 3

$$F_{\text{net from all electrical effects}} = qE + q(v \times B).$$

We will approach this problem in parts, then put everything together at the end.

a.) For the electric field: The electric field vector is $25\mathbf{i}$ nts/C. To determine the force acting on a positive charge, all we have to do is to multiply the electric field vector E by the charge q and we have the electric force qE , direction and all. A negative charge of magnitude q will feel the same size force in the electric field, but the force will be opposite in direction. The easiest way to accommodate this situation is to simply include the sign of the charge in the qE expression (using the sign of a charge as though it has something to do with the direction of a vector is not strictly kosher, but it works, so we'll do it). Doing so yields:

$$\begin{aligned}
 F_{E,\text{fld}} &= qE \\
 &= (-2 \text{ C})(25\mathbf{i} \text{ nt/C}) \\
 &= 50(-\mathbf{i}) \text{ nts.}
 \end{aligned}$$

b.) For the magnetic field: The B -field must have a magnitude and direction such that when it is operated on in the appropriate way (i.e., when we calculate $qv \times B$), the force exerted on the negative charge exactly cancels the force exerted by the electric field (remember, the net acceleration of the particle is zero). As F_E has a direction of $-\mathbf{i}$ (see the calculation in Part a), F_B must have a direction of $+\mathbf{i}$. Including the sign of

the charge in the cross product calculation (the expression $qv \times B$ was defined for the force on a positive charge--as we must reverse the calculated direction when dealing with a negative charge, we might as well effect that sign reversal by including q 's negative sign), we can write:

$$\begin{aligned} F_B &= qv \times B \\ &= (-2 \text{ C})[4 \text{ j m/s}] \times [B \text{ (unit vector) Teslas}]. \end{aligned}$$

From observation, the magnitude of the force cross product is $8B$. From previous arguments, the direction of the cross product must be in the direction of the magnetic force (i.e., in the $+i$ direction). As such:

$$F_B = (8B) (+i).$$

So the question comes down to this: The magnetic field's direction must be such that when j is crossed into it, then multiplied by -1 , the cross product yields a vector in the $+i$ direction. What direction is this?

At the very least, a vector in the $-k$ direction will do the job. For that case, the B-field can be written as $B = B(-k)$.

Note: I say, "At the very least" because an additionally magnetic field component in the direction of motion will not produce a force on the charge ($j \times j = 0$). As that component will not alter anything, the most general way to write out our magnetic field vector would be $B = B_y j - B_z k$.

c.) Putting it all together we get:

$$\begin{aligned} F_{\text{net}} &= q E + q [v \times B] \\ &= (-2 \text{ C})(25i \text{ nt/C}) + (-2 \text{ C})[4(j) \text{ m/s}] \times [B(-k) \text{ T}] \\ &= 50(-i) \text{ nts} + (8B) (+i). \end{aligned}$$

Noting that the acceleration is zero, we can finally write:

$$\begin{aligned} -50i + 8Bi &= 0 \\ \Rightarrow B &= 6.25 \text{ T}. \end{aligned}$$

As a vector, the magnetic field in this problem can be written as:

$$B = 6.25(-k) \text{ teslas}.$$

18.8)

a.) The equation governing the magnitude of B-field generated by a wire is:

$$B = \mu_0 I / (2\pi r),$$

where $\mu_0 = 4\pi \times 10^{-7}$ volt-second/amp (these units are also henrys/meter or tesla-meters/amp), I is the current through the wire, and r the distance from the wire. We want the distance r at which the magnitude of B_{wire} is 6×10^{-5} teslas. Manipulating the above equation we get:

$$\begin{aligned} r &= \frac{(4\pi \times 10^{-7} \text{ T} \cdot \text{m} / \text{A})(8 \text{ A})}{2\pi(6 \times 10^{-5} \text{ T})} \\ &= .0267 \text{ meters.} \end{aligned}$$

b.) If the wire runs North-South (or vice versa), it will produce no B-field IN the N-S direction (remember, B-fields CIRCLE around wires) and, hence, will not add to or subtract from the earth's field. That means the wire must run East-West. If the current runs West (assuming no dip to the earth's field, and remembering the earth's field lines go from the South geographic pole to the North geographic pole), the fields will cancel 2.67 centimeters below the wire. If the current runs East, the fields will cancel 2.67 centimeters above the wire.

18.9) Using the modified right-hand rule, the magnetic field about wires A and D will be counterclockwise and the field around wire C will be clockwise.

a.) The distance between Point P and each wire is the diagonal distance across the square divided by 2. That is,

$$r_p = (.25^2 + .25^2)^{1/2} / 2 = .177 \text{ meters.}$$

With the B-field directions shown in Figure 4, the components add vectorially yielding:

$$\begin{aligned} \mathbf{B} &= B_A [(\cos 45^\circ)\mathbf{i} + (\sin 45^\circ)\mathbf{j}] + \\ &\quad B_D [(\cos 45^\circ)\mathbf{i} - (\sin 45^\circ)\mathbf{j}] + \\ &\quad B_C [(\cos 45^\circ)\mathbf{i} + (\sin 45^\circ)\mathbf{j}]. \end{aligned}$$

As all three magnetic field magnitudes are the same (same current, same distance), we can simplify the above equation to:

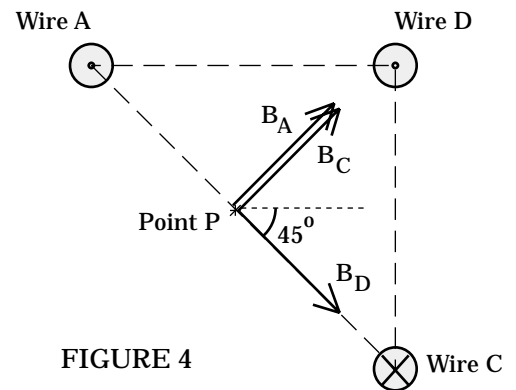


FIGURE 4

$$\begin{aligned}
\mathbf{B} &= B [(3 \cos 45^\circ)\mathbf{i} + (1 \sin 45^\circ)\mathbf{j}] \\
&= [\mu_0 I/(2\pi r)] [2.12 \mathbf{i} + .707\mathbf{j}] \\
&= \left[(4\pi \times 10^{-7} \text{ T}\cdot\text{m/A})(15 \text{ A})/[2\pi(.177 \text{ m})] \right] [2.12 \mathbf{i} + .707\mathbf{j}] \\
&= 3.6 \times 10^{-5} \mathbf{i} + 1.2 \times 10^{-5} \mathbf{j}.
\end{aligned}$$

b.) The magnetic force as a vector is equal to $q\mathbf{v} \times \mathbf{B}$, or:

$$\begin{aligned}
\mathbf{F} &= q \mathbf{v} \times \mathbf{B} \\
&= (-7 \times 10^{-12} \text{ C})[3200 \text{ k m/s}] \times [3.6 \times 10^{-5} \mathbf{i} + 1.2 \times 10^{-5} \mathbf{j}] \\
&= [-2.24 \times 10^{-8} \text{ k m/s}] \times [3.6 \times 10^{-5} \mathbf{i} + 1.2 \times 10^{-5} \mathbf{j}].
\end{aligned}$$

Crossing \mathbf{k} into $(\mathbf{i} + \mathbf{j})$ yields a vector whose direction is $(\mathbf{j} - \mathbf{i})$. Noting that q is negative, the cross product yields:

$$\mathbf{F} = (-8.1 \times 10^{-13} \mathbf{j} + 2.69 \times 10^{-13} \mathbf{i}) \text{ nts.}$$

c.) Along a line between wire A and wire C, the velocity vector will be $3200 \angle -45^\circ$. The unit vector equivalent is $\mathbf{v} = 2262 \mathbf{i} - 2262 \mathbf{j}$ m/s. Doing the cross product in unit vector notation (remember the matrix approach) and keeping in mind that we are dealing with a negative charge, we end up with:

$$\mathbf{F} = (-7.6 \times 10^{-13} \mathbf{k}) \text{ nts.}$$

d.) Along a line between wire C and wire A (this direction is the same as a line from P to A), the velocity vector will be $3200 \angle 135^\circ$. The unit vector equivalent is $\mathbf{v} = -2262 \mathbf{i} + 2262 \mathbf{j}$ m/s. The cross product in that case yields:

$$\mathbf{F} = (7.6 \times 10^{-13} \mathbf{k}) \text{ nts.}$$

e.) The B-field provided by wire A at wire D is $+\mu_0 I/(2\pi r)\mathbf{j}$ while the B-field provided by wire C at wire D is $+\mu_0 I/(2\pi r)\mathbf{i}$. Putting in the numbers yields:

$$\begin{aligned}
\mathbf{B}_{\text{net at D}} &= [\mu_0 I/(2\pi r)] (\mathbf{i} + \mathbf{j}) \\
&= \left[(4\pi \times 10^{-7} \text{ T}\cdot\text{m/A})(15 \text{ A})/[2\pi(.25 \text{ m})] \right] (\mathbf{i} + \mathbf{j}). \\
&= 1.2 \times 10^{-5} (\mathbf{i} + \mathbf{j}).
\end{aligned}$$

The magnetic force on a current-carrying wire is:

$$F = iL \times B,$$

where i is the current in the wire, L is a vector whose magnitude is the length of the wire and whose direction is the direction of current flow, and B is B . The force will therefore be:

$$F = (15 \text{ A})(L \text{ k}) \times (1.2 \times 10^{-5} \text{ i} + 1.2 \times 10^{-5} \text{ j}).$$

The cross product yields:

$$F = 1.8 \times 10^{-4} L \text{ j} - 1.8 \times 10^{-4} L \text{ i}.$$

The force per unit length will, therefore, be:

$$F/L = 1.8 \times 10^{-4} \text{ j} - 1.8 \times 10^{-4} \text{ i}.$$

18.10) The sketch in Figure 5 shows the device assuming **NEGATIVE CHARGE** flows in the circuit (this was Step 1). Note the preponderance of negative charge on the bottom side of the plate making the top side the high voltage side.

Figure 6 shows the device assuming **POSITIVE CHARGE** moves in the circuit (this was Step 2). Note the preponderance of positive charge on the bottom side of the plate making the bottom side the high voltage side.

A voltmeter has both a high voltage and a low voltage terminal. If you attach the high voltage terminal to, say, the bottom of the plate, the meter's needle will swing appropriately if the charge flow is that of positive charge. If, in fact, you were to try this, you would find that the needle would swing in the wrong direction. What does that mean? It means that the high voltage side of the plate must not be the bottom side, which means that the current flow in the circuit is not made up of positive charges.

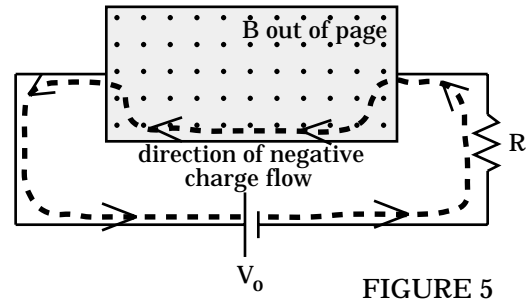


FIGURE 5

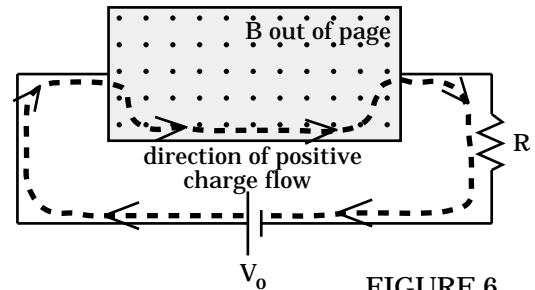


FIGURE 6

18.11) Remembering that full deflection of the galvanometer needle will occur when 5×10^{-4} amps flow through the galvanometer, and assuming that the galvanometer's resistance is 12 ohms, consider:

a.) A voltmeter is designed to measure the voltage difference between two points, usually on either side of a circuit element. If we want a 300 volt (max) voltmeter, we need a meter-circuit such that when 300 volts is placed across it, 5×10^{-4} amps flow through the internal galvanometer causing the galvanometer to register full deflection.

The kind of circuit that will do the trick is shown in Figure 7. Noting that the current is the same for the extra resistor as it is for the galvanometer (the two are in series), we can sum the voltage drops across the individual elements and write:

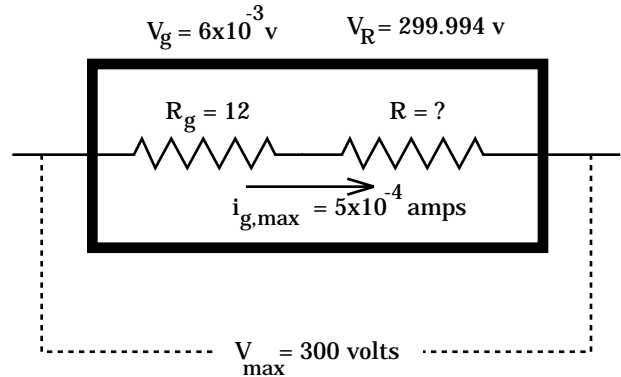


FIGURE 7

$$\begin{aligned}
 V_{\max} &= V_{\text{galv,max}} + V_R \\
 (300 \text{ volts}) &= (i_{\text{galv,max}})(R_{\text{galv}}) + (i_{\text{galv,max}})(R) \\
 (300 \text{ volts}) &= (5 \times 10^{-4} \text{ A})(12 \Omega) + (5 \times 10^{-4} \text{ A}) R \\
 \Rightarrow R &= 6 \times 10^5 \Omega \dots (\text{large as expected})
 \end{aligned}$$

b.) An ammeter is designed to measure the current that passes through it. That means that if we want a .25 amp (max) ammeter, we need a circuit such that when .25 amps pass through it, a current of 5×10^{-4} amps flow through the internal galvanometer. That also means that the rest of the current has to be shunted off elsewhere. That elsewhere is a parallel resistor called a shunt resistor R_{shunt} . Figure 8 shows the circuit. The math follows:

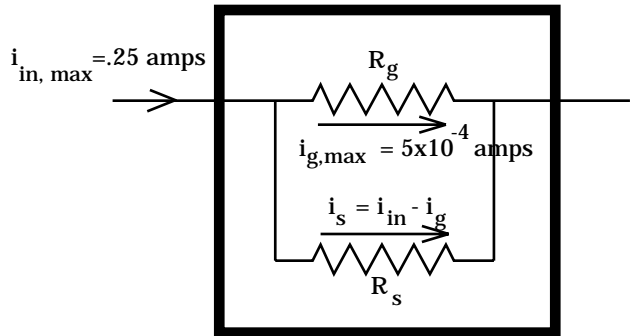


FIGURE 8

As the voltage across a parallel combination is the same for each branch, we can write the following for our .25 amp ammeter:

$$\begin{aligned}
 V_R &= V_{\text{shunt}} \\
 i_{\text{galv,max}} R_g &= (.25 - i_{\text{g,max}}) R_s \\
 (5 \times 10^{-4} \text{ A})(12 \Omega) &= (.2495 \text{ A}) R_s \\
 \Rightarrow R_s &= .024 \Omega \dots (\text{small as expected})
 \end{aligned}$$

18.12) The idea behind Ampere's Law is to get a feel for the direction of the B-field in the region of interest, then find a path over which the magnitude of B is constant. In this case, B circles the long wire. That means that the Amperian path should be a circle.

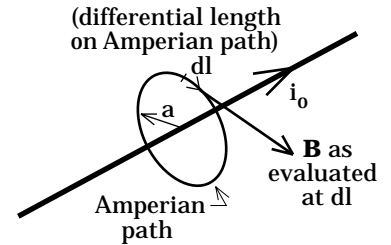


FIGURE 9

a.) To determine the magnitude of B for $r < r_1$:

Figure 9 shows the inner wire and an Amperian path of radius a. Using that sketch, we can write:

$$\begin{aligned}\oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 i_{\text{thru}} \\ \Rightarrow \oint (B)(dl) \cos 0^\circ &= \mu_0 i_0 \\ \Rightarrow B \oint dl &= \mu_0 i_0 \\ \Rightarrow B(2\pi a) &= \mu_0 i_0 \\ \Rightarrow B &= \frac{\mu_0 i_0}{2\pi a}.\end{aligned}$$

If the expression for B is positive, it must circulate as shown in Figure 9 (actually, it's direction depends upon where you are, relative to the wire, but the circulation is as shown).

b.) To determine the magnitude of B for $r_1 < r < r_2$: The sketch to the right shows the inner wire, the cylinder and an Amperian path of radius a for the region between r_1 and r_2 .

Before we can use Ampere's Law, we need to determine the amount of current passing through the Amperian path due to the current flow in the pipe. Given that the current varies as one moves radially out from the wire, we must first determine the current through an arbitrary differential area dA (see Figure 11), then integrate. Note that if the radius of the differential area is c, the thickness of the differential strip is dc and the differential area is $dA = (2\pi c)dc$. As the current density in that differential region is evaluated as $j = (kc)i_0$ (this function was given

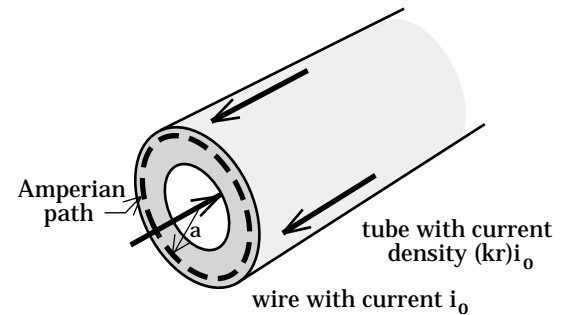


FIGURE 10

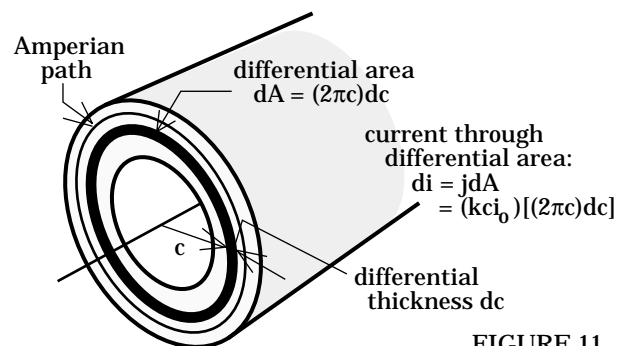


FIGURE 11

in the problem), multiplying the current density by the differential area yields the differential current flowing through the area. Integrating that quantity between $r = r_1$ and $r = a$ gives us the total current through the pipe enclosed by the Amperian path. As the pipe's current and the wire's current are in opposite directions, Ampere's Law becomes:

$$\begin{aligned}
 \oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 i_{\text{thru}} \\
 \Rightarrow \oint (\mathbf{B})(d\mathbf{l}) \cos 0^\circ &= \mu_0 (i_o - \int j dA) \\
 \Rightarrow B \oint d\mathbf{l} &= \mu_0 \left[i_o - \int_{c=r_1}^a (kci_o)(2\pi c) dc \right] \\
 \Rightarrow B(2\pi a) &= \mu_0 i_o \left[1 - 2\pi k \int_{c=r_1}^a c^2 dc \right] \\
 \Rightarrow B &= \frac{\mu_0 i_o \left[1 - \frac{2\pi k}{3} (a^3 - r_1^3) \right]}{2\pi a}.
 \end{aligned}$$

If, after the expression is evaluated, B is found to be positive, it's direction will circulate in a manner similar to that shown in Figure 9. This will correspond to a net resultant current into the page as was the case in that figure. If the sign of B is negative, the circulation will be the other way.

c.) For B outside the complex: The only difference between this part and the situation in Part b is the limits of integration. That is, we now need all the current in the tube (i.e., from $r = r_1$ to $r = r_2$). With an Amperian path of radius $a > r_2$, the final expression becomes:

$$\begin{aligned}
 \oint \mathbf{B} \cdot d\mathbf{l} &= \mu_0 i_{\text{thru}} \\
 \Rightarrow \oint (\mathbf{B})(d\mathbf{l}) \cos 0^\circ &= \mu_0 (i_o - \int j dA) \\
 \Rightarrow B \oint d\mathbf{l} &= \mu_0 \left[i_o - \int_{c=r_1}^{r_2} (kci_o)(2\pi c) dc \right] \\
 \Rightarrow B(2\pi a) &= \mu_0 i_o \left[1 - 2\pi k \int_{c=r_1}^{r_2} c^2 dc \right] \\
 \Rightarrow B &= \frac{\mu_0 i_o \left[1 - \frac{2\pi k}{3} (r_2^3 - r_1^3) \right]}{2\pi a}.
 \end{aligned}$$

d.) For $j = (rk)i_0$ to have the units amps per meter squared, the constant k must have the units meters⁻³.

18.13) Following the hint provided in the problem: Define a differentially wide section of the plate an ARBITRARY distance x units from the origin and call that width dx (see Figure 12).

Note: A common error is to define x to be at a special point like an endpoint. As an example, you might be tempted to make your sketch look like Figure 13 with $x = w/2$ and $r = [(w/2)^2 + y^2]^{1/2}$. This would be a major mistake!!!

With our system variables as defined in Figure 12, treat the current through the differential section (call this di) as though it were current in a single wire. Doing so, determine the magnetic field vector dB due to that current as it manifests itself at Point P.

As a preliminary, determine di : To do this, define a current density function j for the plate, then multiply the current density by the differential area $dA = hdx$.

For the current density function j , we can use macroscopic variables such that:

$$j = \text{current/area} = i_0/[hw]$$

and microscopic variables:

$$j = \text{current/area} = di/[(h)dx].$$

Equating the two yields:

$$di = (i_0/w)dx.$$

With the differential current, we can proceed. To determine the direction of the magnetic field due

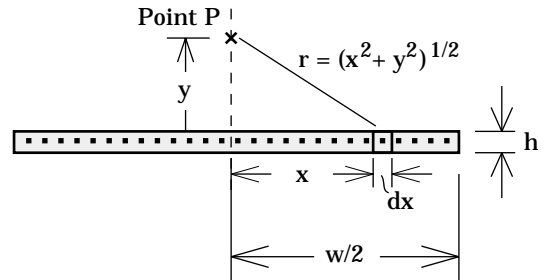


FIGURE 12

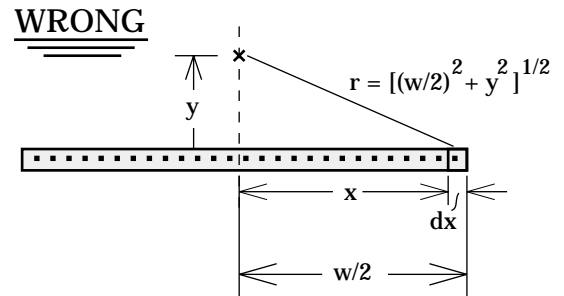


FIGURE 13

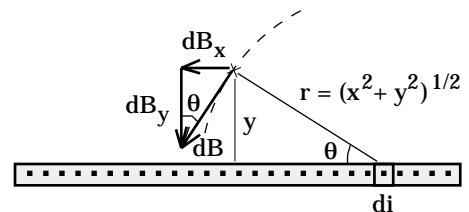


FIGURE 14

to di as evaluated at Point P, draw a circle that passes through Point P and that is centered on the section of wire at x . Using the right-thumb rule, the sense of the B-field is as shown in Figure 14 on the previous page. Note also that that field has been defined as dB . It is put in differential form because the field generated by di is only a small part of the overall field found at Point P.

If we consider the magnetic field produced by a bit of current at $-x$ (see Figure 15), it should be evident that the y components of the two fields will add to zero. As such, the symmetry of the situation allows us to concentrate on the x component only.

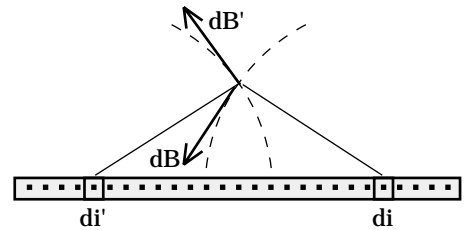


FIGURE 15

The x component of the magnetic field will, according to the sketch in Figure 14 on the previous page, be $dB_x = dB \sin\theta$. From the

geometry presented in that sketch, $\sin \theta = y/r$ (you should convince yourself that the two angles defined as θ in Figure 14 are, indeed, the same).

The magnitude of the differential field dB at Point P will be:

$$dB = \frac{\mu_o (di)}{2\pi r},$$

where all the variables have been previously defined (if you hadn't known the expression for the B-field due to a long wire, you would have had to have derived it using Ampere's Law). The x component of that field will be:

$$dB_x = \frac{\mu_o (di)}{2\pi r} \sin \theta.$$

Putting in all the variables (i.e., di , r , etc.), we get:

$$\begin{aligned} dB_x &= \frac{\mu_o (di)}{2\pi r} \sin \theta \\ &= \frac{\mu_o [j(dA)] \left[\frac{y}{r} \right]}{2\pi r} \\ &= \frac{\mu_o \left[\frac{i_o dx}{w} \right]}{2\pi \left[(x^2 + y^2)^{1/2} \right]} \left[\frac{y}{(x^2 + y^2)^{1/2}} \right] \\ &= \frac{\mu_o y i_o}{2\pi w (x^2 + y^2)} dx. \end{aligned}$$

The total magnetic field at Point P will be the sum all the differential fields. Using integration (note the limits change halfway down), this is:

$$\begin{aligned} \mathbf{B} &= \int d\mathbf{B}_x \\ &= \frac{\mu_0 y i_0}{2\pi w} \int_{x=-w/2}^{w/2} \left[\frac{1}{(x^2 + y^2)} \right] dx \\ &= \frac{\mu_0 y i_0}{2\pi w} (2) \int_{x=0}^{w/2} \left[\frac{1}{(1x^2 + 0x + 1y^2)} \right] dx. \end{aligned}$$

Integrals of the form $\int \left[\frac{1}{ax^2 + bx + c} \right] dx = \frac{2}{(4ac - b^2)^{1/2}} \tan^{-1} \left[\frac{2ax + b}{(4ac - b^2)^{1/2}} \right]$ as

long as $4ac > b^2$. For our integral, $a = 1$, $b = 0$, and $c = y^2$.

Although it is not particularly important, the evaluation of this integral for our situation yields a magnetic field vector (unit vector and all) of:

$$\mathbf{B} = \frac{\mu_0 i_0}{\pi w} \tan^{-1} \left[\frac{w}{2y} \right] (-\mathbf{i}).$$

18.14) Defined in Figure 16 is a sketch of our system with a differential length of current-carrying wire $d\mathbf{l} = dx$ ($d\mathbf{l}$ is the variable traditionally used in Biot Savart, whereas dx is the appropriate variable for our particular geometry), a vector $\mathbf{r} = (x^2 + y^2)^{1/2}$ from dx to the point in question at $(0, y)$, and the angle θ between dx and \mathbf{r} .

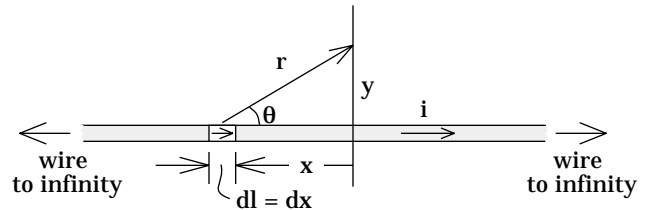


FIGURE 16

Noting that the $\sin \theta = y/r$, we can write Biot Savart as:

$$\begin{aligned} dB &= \frac{\mu_0 i}{4\pi} \frac{(dl) \sin \theta}{r^2} \\ &= \frac{\mu_0 i}{4\pi} \frac{(dx) \left(\frac{y}{r} \right)}{r^2} \\ &= \frac{\mu_0 i y}{4\pi} \frac{(dx)}{(x^2 + y^2)^{3/2}}. \end{aligned}$$

This is the differential magnetic field at the coordinates (0, y) due to the current flowing in the differential section of wire $dl = dx$ located at $-x$. To get the total magnetic field at (0, y), we must either sum up all the differential magnetic field quantities due to all the wire sections from $-\infty$ to $+\infty$, or do a similar integration between 0 and $+\infty$ and double the solution. We will do the latter:

$$\begin{aligned}
 B &= \int dB \\
 &= 2 \left[\frac{\mu_0 i y}{4\pi} \int_{x=0}^{\infty} \frac{(dx)}{(x^2 + y^2)^{3/2}} \right] \\
 &= \frac{\mu_0 i y}{2\pi} \left[\frac{1}{y^2} \frac{x}{(x^2 + y^2)^{1/2}} \right]_{x=0}^{\infty} \\
 &= \frac{\mu_0 i y}{2\pi y^2} \left[\frac{\infty}{(\infty^2 + y^2)^{1/2}} - \frac{0}{(0^2 + y^2)^{1/2}} \right].
 \end{aligned}$$

On the surface, this doesn't look good. Upon closer inspection, things work out nicely. How so? Because for very, very large x :

$$\frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{(x^2)^{1/2}} = 1.$$

As infinity is as large an x as is possible, we can simplify our expression to:

$$\begin{aligned}
 B &= \frac{\mu_0 i y}{2\pi y^2} [1 - 0] \\
 &= \frac{\mu_0 i}{2\pi y}.
 \end{aligned}$$

This is the same expression we derived with considerably less difficulty using Ampere's Law.